Chapter 4.5: Indeterminate Forms and L'Hôpital's Rule

L'Hôpital's Rule

Let f and g be continuous.
Problem:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

Solution:

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

Example:

$$\lim_{x\to 0} \frac{3x - \sin(x)}{x} = \lim_{x\to 0} \frac{3 - \cos(x)}{1} = 2$$

Note: $\frac{f'(x)}{g'(x)}$ is NOT $\frac{d}{dx}\frac{f(x)}{g(x)}$. No quotient rule needed.

Examples

- $\lim_{x\to 0} \frac{1-\cos(x)}{x+x^2} \overset{H}{=} \lim_{x\to 0} \frac{\sin(x)}{1+2x} = \frac{0}{1} = 0$

  Note $\lim_{x\to 0} \frac{\sin(x)}{1+2x} \overset{H}{=} \lim_{x\to 0} \frac{\cos(x)}{2} = \frac{1}{2}$ is WRONG!!
- $\lim_{t\to 0} \frac{\sin(t^2)}{t} \overset{H}{=} \lim_{t\to 0} \frac{\cos(t^2)\cdot 2t}{1} = 0$
- $\lim_{\theta\to 0} \frac{\cos(\theta)-1}{e^\theta-\theta-1} \overset{H}{=} \lim_{\theta\to 0} \frac{-\sin(\theta)}{e^\theta-1} \overset{H}{=} \lim_{\theta\to 0} \frac{-\cos(\theta)}{e^\theta} = -1$
- $\lim_{x\to\infty} \frac{x^2}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$
- $\lim_{x\to\infty} x\sin(1/x)$

  $= \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} \overset{H}{=} \lim_{x\to\infty} \frac{\cos(1/x)\cdot(-1/x^2)}{-1/x^2} = \lim_{x\to\infty} \cos(1/x) = \cos(0) = 1.$
- $\lim_{x\to 0} \frac{1}{x} - \frac{1}{\sin(x)} = \lim_{x\to 0} \frac{\sin(x)}{x\sin(x)} - \frac{x}{x\sin(x)} = \lim_{x\to 0} \frac{\sin(x)-x}{x\sin(x)} \overset{H}{=}$

  $\lim_{x\to 0} \frac{\cos(x)-1}{\sin(x)+x\cos(x)} \overset{H}{=} \lim_{x\to 0} \frac{-\sin(x)}{\cos(x)+\cos(x)-x\sin(x)} = \frac{0}{2} = 0.$

Indeterminate Powers

How to deal with limits looking like 1^∞, 0^0, or ∞^0?

Example: $\lim_{x\to 0} x^x$

We use $L = \lim_{x\to 0} x^x$. First we take ln of both sides.

$$\ln(L) = \ln\left(\lim_{x\to 0} x^x\right) = \lim_{x\to 0} \ln(x^x) = \lim_{x\to 0} x\ln(x) = \lim_{x\to 0} \frac{\ln(x)}{1/x}$$
$$\stackrel{H}{=} \lim_{x\to 0} \frac{1/x}{-1/x^2} = \lim_{x\to 0} -x = 0$$

So we got

$$\ln(L) = 0$$
$$L = 1$$

and the final answer is $\lim_{x\to 0} x^x = 1$.

Examples

1. $\lim_{x\to 0}(1+x)^{1/x} = e$
   Set $L = \lim(1+x)^{1/x}$.
   $\ln(L) = \ln\left(\lim_{x\to 0}(1+x)^{1/x}\right) = \lim_{x\to 0}\ln\left((1+x)^{1/x}\right) = \lim_{x\to 0}\frac{\ln(1+x)}{x} \overset{H}{=}$
   $\lim_{x\to 0}\frac{\frac{1}{1+x}}{1} = 1$. Hence $L = e^1 = e$.
2. $\lim_{x\to\infty} x^{1/x} = 1$
   Set $L = \lim x^{1/x}$
   $\ln(L) = \ln\left(\lim_{x\to\infty} x^{1/x}\right) = \lim_{x\to\infty}\ln\left(x^{1/x}\right) = \lim_{x\to\infty}\frac{\ln(x)}{x} \overset{H}{=} \lim_{x\to\infty}\frac{1/x}{1} =$
   $\lim_{x\to\infty}\frac{1}{x} = 0$ Hence $L = e^0 = 1$.
3. $\lim_{x\to 0}(\cos(x))^{1/x^2} = e^{-1/2} = \frac{1}{\sqrt{e}}$
   $\ln(L) = \ln\left(\lim_{x\to 0}(\cos(x))^{1/x^2}\right) = \lim_{x\to 0}\ln\left((\cos(x))^{1/x^2}\right) = \lim_{x\to 0}\frac{\ln(\cos(x))}{x^2} \overset{H}{=}$
   $\lim_{x\to 0}\frac{(1/\cos(x))\cdot(-\sin(x))}{2x} = \lim_{x\to 0}\frac{-\tan(x)}{2x} \overset{H}{=} \lim_{x\to 0}\frac{-\sec^2(x)}{2} = -\frac{1}{2}$

L'Hôpital's Trap

Compute the following limit with and without L'Hôspital's rule.

$$\lim_{x\to\infty} \frac{x+\sin x}{x} = \lim_{x\to\infty} \frac{1+\cos x}{1} = \text{ does not exist}$$

$$\lim_{x\to\infty} \frac{x+\sin x}{x} = \lim_{x\to\infty} \frac{x}{x} + \frac{\sin x}{x} = 1 + 0 = 1$$

Notice that L'Hôspital's rule is $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ if the right-hand side exists!

From an old exam

Compute $\lim_{x\to\infty} (\pi + 4x)^{\frac{1}{\ln(x)}}$

$$
\begin{aligned}
L &= \lim_{x\to\infty} (\pi + 4x)^{\frac{1}{\ln(x)}} \\
\ln L &= \ln \lim_{x\to\infty} (\pi + 4x)^{\frac{1}{\ln(x)}} \\
&= \lim_{x\to\infty} \ln\left((\pi + 4x)^{\frac{1}{\ln(x)}}\right) \\
&= \lim_{x\to\infty} \frac{\ln(\pi + 4x)}{\ln(x)} \to \frac{\infty}{\infty} \\
&\overset{H}{=} \lim_{x\to\infty} \frac{\frac{4}{\pi+4x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{4x}{\pi + 4x} \to \frac{\infty}{\infty} \\
&\overset{H}{=} \lim_{x\to\infty} \frac{4}{4} = 1 \\
\ln L &= 1 \\
L &= e^1 = e
\end{aligned}
$$